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A CONJECTURE OF AX AND DEGENERATIONS OF FANO VARIETIES

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ABSTRACT

James Ax conjectured that every pseudo algebraically closed field is C_1. We prove this conjecture in characteristic 0 by relating it to degenerations of Fano varieties.

A field k is called C_1 if every homogeneous form $f(x_0,\dots,x_n)\in k[x_0,\dots,x_n]$ of degree $\le n$ has a nontrivial zero. Examples of C_1 fields are finite fields (Chevalley) and function fields of curves over an algebraically closed field (Tsen).

A field is called *PAC* (pseudo algebraically closed) if every geometrically integral k-variety has a k-point. A k-variety X is called geometrically integral if $X\times_k \bar{k}$ is integral (that is, irreducible and reduced) where $\bar{k}$ is an algebraic closure of k. Equivalently, in the terminology of Weil, X is an absolutely irreducible variety defined over k.

PAC fields were introduced in [Ax68]; see [FJ05] for an exhaustive and up to date treatment.

The aim of this paper is to prove in characteristic 0 a conjecture of Ax, posed in [Ax68, Problem 3].

Theorem 1: *Every PAC field of characteristic 0 is C_1.*

[Ax68, Theorem D] proves this for fields whose absolute Galois group is abelian and [FJ05, 21.3.6(a)] settles the case of fields that contain an algebraically closed subfield.

Following an idea of [DJL83], we deduce Theorem 1 from the next result which holds for all fields of characteristic zero.

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THEOREM 2: *Let k be a field of characteristic 0 and $f_1, \dots, f_s \in k[x_0, \dots, x_n]$ homogeneous polynomials such that $\sum_i \deg f_i \le n$. Let*

$$X = X(f_1, \dots, f_s) := (f_1 = \dots = f_s = 0) \subset \mathbb{P}^n_k$$

be the subscheme they define in projective n-space. Then

(1) *X contains a geometrically irreducible k-subvariety $Y \subset X$.*
(2) *If k is PAC then X has a k-point.*

If k is PAC, then Y has a k-point which is also a k-point of X, thus (1) implies (2). The case $s = 1$ of Theorem 2 (2) is precisely Theorem 1. The more general version proved here is sometimes called property C_1'.

In order to prove Theorem 2, we represent (a subscheme of) the scheme $X(f_1, \dots, f_s)$ as a special fiber of a family $Z \to \mathbb{P}^1$ over the projective line whose general fiber is a smooth hypersurface or a complete intersection variety. The restrictions on the degree are equivalent to assuming that the canonical class of the general fiber of $Z \to \mathbb{P}^1$ is negative. This approach raises further interesting questions about degenerations of Fano varieties, we discuss these in Question 16.

It is thus sufficient to prove the following more general result.

THEOREM 3: *Let k be a field of characteristic 0, C a smooth k-curve, Z a reduced, irreducible, projective k-variety and $g\colon Z \to C$ a morphism. Assume that the generic fiber F_{gen} is*

(1) *smooth,*
(2) *geometrically irreducible, and*
(3) *Fano (that is, $-K_{F_{gen}}$ is ample).*

Let $c \in C$ be a closed point with residue field $k(c)$. Then the fiber $g^{-1}(c)$ contains a $k(c)$-subvariety which is geometrically irreducible. If, in addition, every $k(c)$-irreducible component of $g^{-1}(c)$ is smooth (or normal), then $g^{-1}(c)$ contains a $k(c)$-irreducible component which is geometrically irreducible.

If Z is smooth and every geometric fiber of g is a simple normal crossing divisor Definition 7, then by the main theorem of [GHS03], $g^{-1}(c)$ always contains an irreducible component which has multiplicity 1 in $g^{-1}(c)$. The next example shows that in general none of these are geometrically irreducible.

Example 4: Let $k = \mathbb{Q}$, $C = \mathbb{P}^1_{s:t}$ and in $\mathbb{P}^1_{s:t} \times \mathbb{P}^2_{u:v:w}$ consider the family of conics

$$Z := (t(u^2 + v^2) + sw^2 = 0) \subset \mathbb{P}^1_{s:t} \times \mathbb{P}^2_{u:v:w},$$

with projection $g\colon Z \to C$. Set $c := (0:1)$. Then $g^{-1}(c)$ is a pair of lines which are conjugate over $\mathbb{Q}$. The only $k(c)$-subvariety of $g^{-1}(c)$ which is geometrically irreducible is the point $P := (0:1) \times (0:0:1)$. Note that every geometric irreducible component of $g^{-1}(c)$ is smooth, but $g^{-1}(c)$ itself is irreducible and singular.

If we blow up P, we get $g' : Z' := B_P Z \to C$ with exceptional curve $E \subset Z'$. All the assumptions of Theorem 3 are now satisfied, and $E \subset {g'}^{-1}(c)$ is the unique $k(c)$-irreducible component of ${g'}^{-1}(c)$ which is geometrically irreducible.

By explicit computation, E has multiplicity 2 in ${g'}^{-1}(c)$.

5 Proof of Theorem 3 ⇒ Theorem 2.

If $\deg f_i = 1$ for every i then X is a nonempty linear subspace of $\mathbb{P}^n_k$ hence geometrically irreducible. In all other cases $s \le n-1$.

Let V_i denote the affine space of homogeneous polynomials in $k[x_0, \ldots, x_n]$ whose degree equals $\deg f_i$. As explained in [Har77, Exercise II.8.4], there is a Zariski open set $W \subset \prod_i V_i$ such that if $(h_1, \ldots, h_s) \in W(\bar{k})$ then the complete intersection variety $(h_1 = \cdots = h_s = 0) \subset \mathbb{P}^n$ is smooth and geometrically irreducible of dimension $n-s \ge 1$. Since k is infinite, we can choose $h_1, \ldots, h_s \in k[x_0, \ldots, x_n]$.

Let $Z_1 \subset \mathbb{P}^n_k \times \mathbb{P}^1_k$ be defined by the equations

$$(uf_1 + vh_1 = \cdots = uf_s + vh_s = 0) \subset \mathbb{P}^n_k \times \mathbb{P}^1_k,$$

where $(u : v)$ are the coordinates on the projective line $\mathbb{P}^1$. The fiber of the projection $g_1\colon Z_1 \to \mathbb{P}^1$ over $(0:1)$ is smooth and geometrically irreducible, thus this holds for all points in an open subset of $\mathbb{P}^1$. Thus there is a unique irreducible component $Z \subset Z_1$ which dominates $C := \mathbb{P}^1$ with projection $g\colon Z \to C$. Furthermore, general fibers of g are smooth and geometrically irreducible. By [Har77, Exercise II.8.4] the canonical sheaf of these fibers is the restriction of $\mathcal{O}_{\mathbb{P}^n}(\sum \deg f_i - n - 1)$, which is negative by assumption. Thus by Theorem 3, the fiber of $g\colon Z \to C$ over $(1:0)$ contains a geometrically irreducible k-subvariety Y which is also a k-subvariety of X. ∎

The proof of Theorem 3 proceeds in two steps. First we use resolution of singularities $h\colon Y \to Z$ to reduce to the case $g \circ h\colon Y \to C$ where every fiber is a simple normal crossing divisor (see Definition 7).

Then we apply a variant of the Kollár–Shokurov Connectedness Theorem [Kol92, Theorem 17.4] to a carefully chosen auxiliary $\mathbb{Q}$-divisor D to prove that every fiber contains a geometrically irreducible component. Connectedness enters through the following elementary observation (cf. [Har77, Remark III.7.9.1]).

(∗) Let X_k be a smooth variety. Then X is geometrically irreducible iff $X_{\bar{k}}$ is connected.

A further complication is that one of the assumptions of [Kol92, 17.4] is not satisfied in our case, but this is compensated by other special features of the current situation. Thus I go through the whole proof in paragraph 9 following an informal explanation of the Connectedness Theorem. See paragraph 7 for the definitions and conventions concerning divisors and the associated sequences of sheaves.

6 Introduction to the Connectedness Theorem. Let Y be a smooth projective variety over an algebraically closed field $\bar{k}$ of characteristic 0 such that $-K_Y$ is ample. It can happen that there is an effective divisor A such that

$$-A \sim K_Y + (\text{ample divisor}). \tag{6.1}$$

We claim that this implies that $\operatorname{Supp} A$ is connected. Indeed, consider the exact sequence (cf. (7.4))

$$0 \to \mathcal{O}_Y(-A) \to \mathcal{O}_Y \to \mathcal{O}_A \to 0,$$

and its associated cohomology sequence

$$\bar{k} \cong H^0(Y, \mathcal{O}_Y) \to H^0(A, \mathcal{O}_A) \to H^1(Y, \mathcal{O}_Y(-A)).$$

By Kodaira's vanishing theorem (see, for instance, [GH78, p. 154] or [KM98, Section 2.5]),

$$H^1(Y, \mathcal{O}_Y(-A)) = H^1(Y, \mathcal{O}_Y(K_Y + (\text{ample divisor}))) = 0,$$

thus we conclude that $H^0(A, \mathcal{O}_A) \cong \bar{k}$. This implies that $\operatorname{Supp} A$ is connected, since otherwise we would have sections which are constant on each connected component of A but not globally constant.

This seems nice, but in practice it rarely gives anything interesting. In order to get a more useful result, consider the case when we can write

$$B - A \sim K_Y + (\text{ample divisor}), \tag{6.2}$$

where A, B are effective with no common irreducible components. Then, as in (7.5), $B|_A$ is also a Cartier divisor and we get an exact sequence

$$0 \to \mathcal{O}_Y(B - A) \to \mathcal{O}_Y(B) \to \mathcal{O}_A(B|_A) \to 0,$$

and its associated exact cohomology sequence

$$H^0(Y, \mathcal{O}_Y(B)) \to H^0(A, \mathcal{O}_A(B|_A)) \to H^1(Y, \mathcal{O}_Y(B-A)) = 0,$$

where the vanishing is again by Kodaira's theorem. As before, using the obvious inclusion $H^0(A, \mathcal{O}_A) \hookrightarrow H^0(A, \mathcal{O}_A(B|_A))$, we conclude that if $H^0(Y, \mathcal{O}_Y(B)) \cong \bar{k}$ then $\operatorname{Supp} A$ is connected.

Let us try to apply this to the case $s = 1$ of Theorem 2. It is very important that, for the rest of this discussion, all varieties and divisors will be defined over k, but we apply the Connectedness Theorem over $\bar{k}$.

Using the notation of paragraph 5, $Z \subset \mathbb{P}^n \times \mathbb{P}^1$ is a hypersurface and let $h: Y \to Z$ be a resolution of singularities. Let $F_0 \subset Y$ be the fiber of $g \circ h$ over the point $(1:0)$ and assume that h is an isomorphism outside F_0 and F_0 is a simple normal crossing divisor Definition 7. Pulling back K_Z gives an identity

$$K_Y \sim h^*K_Z + E, \quad \text{where } E \text{ is } h\text{-exceptional.}$$

Write $E = A - B$ where A, B are effective with no common irreducible components and $\operatorname{Supp}(A+B) \subset \operatorname{Supp} F_0$. We can rearrange the above linear equivalence as

$$B - A \sim K_Y + h^*(-K_Z). \tag{6.3}$$

We aim to apply the previous argument to prove that $\operatorname{Supp} A$ is geometrically connected.

The good news is that B is h-exceptional, thus $H^0(Y, \mathcal{O}_Y(B)) \cong \bar{k}$ is automatic. (Indeed, any function with poles only along B gives a function on the normalization of Z with poles in the codimension ≥ 2 set which is the preimage of $h(B)$, hence constant.)

The bad news is that although $-K_Z$ is ample on the fibers of $g: Z \to \mathbb{P}^1$, it is not ample on Z. This leads to a technical complication: instead of taking H^0 we have to use $(g \circ h)_*$ and the Kodaira vanishing theorem needs to be replaced by the vanishing of a certain $R^1(g \circ h)_*$ (see Theorem 8). A further problem is that $h^*(-K_Z)$ is not even ample on F_0, but this again turns out to be a small difficulty. While these make everything technically harder, the advantage is that we now care about K_Y, A, B only in a neighborhood of F_0.

Since F_0 is a simple normal crossing divisor, every k-irreducible component of A is smooth. If by accident A is k-irreducible, then A itself is smooth and geometrically connected, hence geometrically irreducible.

We have no reason to expect A to be k-irreducible, in fact, A may even be empty. Thus we try to modify A and B.

Since any two fibers of $g \circ h$ are linearly equivalent, subtracting a multiple λF_0 from E is like subtracting any other fiber, thus, in a neighborhood of F_0, we still have a linear equivalence

$$K_Y \sim h^* K_Z + E - \lambda F_0.$$

Write $E - \lambda F_0 = \sum a_i(\lambda) P_i$ where the $a_i(\lambda)$ are linear functions in λ and the P_i are distinct k-irreducible divisors. Note that this makes sense for any $\lambda \in \mathbb{Q}$ (and even for $\lambda \in \mathbb{R}$), as long as we use formal linear combinations of divisors with real or rational coefficients.

There is a unique choice λ_0 such that $\max_i\{a_i(\lambda_0)\} = 1$. If λ_0 is an integer then we are dealing with actual divisors, but in general λ_0 is rational and we are inevitably led to $\mathbb{Q}$-divisors. Set

$$I := \{i : a_i(\lambda_0) = 1\} \quad \text{and} \quad J := \{i : a_i(\lambda_0) \notin \mathbb{Z}\}.$$

Every rational number can be written uniquely as an integer plus a nonnegative rational < 1. Correspondingly, we can write

$$E - \lambda_0 F_0 = A - B + \Delta$$

where $A = \sum_{i \in I} P_i$ (all coefficients $= 1$!), B is an effective divisor with integer coefficients, $\operatorname{Supp} A$ and $\operatorname{Supp} B$ have no common irreducible components and $\Delta = \sum_{j \in J} \alpha_j Q_j$ is a $\mathbb{Q}$-divisor such that $0 < \alpha_j < 1$ with the Q_j distinct.

Note further that

$$B - A \sim K_Y + h^*(-K_Z) + \Delta, \tag{6.4}$$

which closely resembles (6.3).

Thus we need to use the generalized Kodaira vanishing theorem (see Theorem 8), which roughly says that the vanishing of H^1 still holds for divisors of the form

$$K_Y + (\text{ample divisor}) + \Delta,$$

where Δ is a simple normal crossing divisor with all coefficients between 0 and 1.

We hope to prove that P_i is geometrically connected for each $i \in I$. To see this, write $A_i' := \sum_{j \in I \setminus \{i\}} P_j$. We apply our usual argument to

$$B - P_i \sim K_Y + (h^*(-K_Z) + \epsilon A_i') + (\Delta + (1 - \epsilon) A_i') \tag{6.5}$$

for some $0 < \epsilon \ll 1$.

If L is an ample divisor and D is any divisor then $A + \epsilon D$ is also ample for $0 < \epsilon \ll 1$. Unfortunately, $h^*(-K_Z)$ is only semi-ample (that is, the pull back of an ample divisor) and $h^*(-K_Z) + \epsilon A'_i$ is not even semi-ample.

We have, however, considerable freedom to fiddle with the equation (6.5) since we can add (or subtract) small multiples of irreducible components of Δ without changing the assumption that $0 < \alpha_i < 1$ for every i. This process, informally known as **tie breaking**, was introduced in an unpublished preprint of Reid [Rei83]. The end result is that after careful choices we can write

$$B - P_i \sim K_Y + (h^*(-K_Z) + \Delta_1) + \Delta_2 \tag{6.6}$$

where $h^*(-K_Z) + \Delta_1$ is ample on the fibers of $g \circ h$ and Δ_2 is a simple normal crossing divisor with all coefficients between 0 and 1. Thus we get that P_i is geometrically irreducible, as required.

Definition 7: Let Y be a smooth k-variety. A prime divisor P is an irreducible and reduced codimension 1 subvariety. A $\mathbb{Q}$-divisor is a formal linear combination

$$D = \sum a_i P_i \quad \text{where } a_i \in \mathbb{Q}, \tag{7.1}$$

and the P_i are prime divisors. Note that the P_i are prime divisors over k but they may have several irreducible components over $\bar{k}$.

D is called a **simple normal crossing** divisor if the P_i are smooth and $D \times_k \bar{k}$ is a divisor with normal crossings as in [Har77, p. 391]. In particular, $(x^2+y^2=0) \subset \mathbb{A}^2_{\mathbb{R}}$ is not a simple normal crossing divisor but $(x^2+y^2=0) \subset \mathbb{A}^2_{\mathbb{C}}$ is one.

Write $D = \sum a_i P_i$ with the P_i distinct. The support of D is $\operatorname{Supp} D := \bigcup_{i: a_i \neq 0} P_i$. D is called effective if $a_i \geq 0$ for every i. Set

$$D_{\geq 1} := \sum_{i: a_i \geq 1} a_i P_i. \tag{7.2}$$

Note that in characteristic 0

$$(D_{\geq 1}) \times_k \bar{k} = (D \times_k \bar{k})_{\geq 1}. \tag{7.3}$$

A $\mathbb{Q}$-divisor D is called ample if mD is an ample divisor for some (or all) $m > 0$ such that ma_i is an integer for every i.

Two $\mathbb{Q}$-divisors D_1, D_2 are called $\mathbb{Q}$-linearly equivalent (denoted by $D_1 \sim_{\mathbb{Q}} D_2$) if there is an integer $m > 0$ such that mD_1 and mD_2 are linearly equivalent integral divisors. (Note that even if the D_i are integral divisors, $\mathbb{Q}$-linear equivalence is slightly different from linear equivalence if $\operatorname{Pic}(Y)$ contains torsion classes.)

Let $D = \sum m_i P_i$ be an integral divisor. Then $\mathcal{O}_Y(D)$ denotes the sheaf of rational functions on Y which have a pole of order at most m_i along P_i for $m_i > 0$ and a zero of order at least $-m_i$ along P_i if $m_i < 0$. Since Y is smooth, every integral divisor is Cartier, thus $\mathcal{O}_Y(D)$ coincides with $\mathcal{L}(D)$ defined in [Har77, Section II.6].

If D is effective then for $\mathcal{O}_Y(D)$ we do not require any vanishing, thus we have an injection $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(D)$. Dually, for $\mathcal{O}_Y(-D)$ we do not allow any poles, thus we have an injection $\mathcal{O}_Y(-D) \hookrightarrow \mathcal{O}_Y$. The quotient sheaf is the structure sheaf of a subscheme which, somewhat sloppily, is also denoted by D [Har77, Proposition II.6.18]. This gives the basic exact sequence

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0. \tag{7.4}$$

Let E be any integral divisor. We can tensor the above sequence by $\mathcal{O}_Y(E)$. Since tensoring by a locally free sheaf is exact, we get another exact sequence

$$0 \to \mathcal{O}_Y(E - D) \to \mathcal{O}_Y(E) \to \mathcal{O}_Y(E) \otimes \mathcal{O}_D \to 0. \tag{7.5}$$

Let $\{\phi_i\}$ be the local defining equations of E as a Cartier divisor. If $\operatorname{Supp} E$ does not contain any of the irreducible components of $\operatorname{Supp} D$, then $\{\phi_i|_D\}$ define a Cartier divisor $E|_D$ and we have an isomorphism

$$\mathcal{O}_Y(E) \otimes \mathcal{O}_D \cong \mathcal{L}(E|_D) =: \mathcal{O}_D(E|_D).$$

(Here we used the fact that in a regular local ring R, if $f, g \in R$ have no common irreducible factors then f is not a zero divisor in R/g, cf. [Har77, pp. 184–5].) Note that in general $\mathcal{O}_D$ has nilpotents, so $\mathcal{L}(E|_D)$ needs the scheme theoretic definition in [Har77, Section II.6] and $\mathcal{O}_D(E|_D)$ is just my notation for the same sheaf. If E is also effective and $\operatorname{Supp} E$ does not contain any of the irreducible components of $\operatorname{Supp} D$, then $E|_D$ is also an effective Cartier divisor, thus we again have an injection

$$\mathcal{O}_D \hookrightarrow \mathcal{O}_D(E|_D). \tag{7.6}$$

The canonical sheaf ω_Y of Y is defined in [Har77, Section II.8]. Since ω_Y is an invertible sheaf on Y, it can be written as $\omega_Y \cong \mathcal{O}_Y(K)$ for some divisor

K. The divisor K is not unique, but its linear equivalence class is. This linear equivalence class is denoted by K_Y and called the canonical class of Y.

We use the following generalization of the Kodaira vanishing theorem. (See [KM98, Section 2.5] for a relatively short proof and for further references. Note that there is a misprint in the relevant Corollary 2.68. In the last line $\omega_Y \otimes M$ should be $\omega_Y \otimes L$.)

Theorem 8 (Kawamata–Viehweg vanishing): *Let Y be a smooth, projective variety over a field of characteristic 0, W any variety and $f\colon Y \to W$ a morphism. Let M and $\Delta = \sum a_i P_i$ be $\mathbb{Q}$-divisors on Y with the P_i distinct and L an integral divisor on Y with the following properties:*

(1) *M is ample;*
(2) *Δ is a simple normal crossing divisor and $0 < a_i < 1$ for every i;*
(3) *$L \sim_{\mathbb{Q}} M + \Delta$.*

Then $R^i f_*(\mathcal{O}_Y(K_Y + L)) = 0$ for $i \geq 1$.

I want to stress that even in the special case when the general fiber of f is a smooth hypersurface (which is all one needs for Theorem 1) the flexibility provided by $\mathbb{Q}$-divisors is crucial.

Now we can prove the key technical result of this paper.

Theorem 9 (Connectedness theorem): *Let Y be an irreducible, smooth, projective variety over a field of characteristic 0, C a smooth projective curve and $f\colon Y \to C$ a morphism with geometrically connected fibers. Let $D = \sum a_i P_i$ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y such that*

(1) *D is a simple normal crossing divisor,*
(2) *D is f-vertical (that is, its support is contained in the union of finitely many fibers of f), and*
(3) *$-(K_Y + D)$ is ample.*

Then every fiber of $f\colon \operatorname{Supp} D_{\geq 1} \to C$ is geometrically connected.

Proof: The conclusion is geometric by (7.3), thus we may assume that we are over an algebraically closed field.

Write $D = A - B + \Delta$ where A, B have positive integer coefficients and without common irreducible components, Δ is effective and if we write $\Delta = \sum a_i P_i$ with the P_i distinct then $0 \leq a_i < 1$ for every i. Note that $\operatorname{Supp} A = \operatorname{Supp} D_{\geq 1}$ and A, B are both f-vertical.

Claim: *Notation as above. Then*

(4) *if* $A \neq 0$ *then* $f_*\mathcal{O}_A(B|_A)$ *is a quotient of* $f_*\mathcal{O}_Y(B)$.

Proof: Consider the exact sequence

$$0 \to \mathcal{O}_Y(B - A) \to \mathcal{O}_Y(B) \to \mathcal{O}_A(B|_A) \to 0,$$

and apply f_* to get the exact sequence of coherent sheaves on C

$$f_*\mathcal{O}_Y(B) \to f_*\mathcal{O}_A(B|_A) \to R^1f_*\mathcal{O}_Y(B - A).$$

Observe that

$$B - A = -D + \Delta \sim_{\mathbb{Q}} K_Y + (-(K_Y + D)) + \Delta.$$

(It is best not to think of the last $\sim_{\mathbb{Q}}$ as an equality since K_Y is not a well-defined divisor, only a linear equivalence class of divisors.)

By Theorem 8 we conclude that $R^1f_*\mathcal{O}_Y(B - A) = 0$ and so $f_*\mathcal{O}_A(B|_A)$ is the quotient of the sheaf $f_*\mathcal{O}_Y(B)$. ∎

There are two further key points.

Claim: *Notation as above. Then*

(5) $f_*\mathcal{O}_Y(B)$ *is a rank 1 locally free sheaf on* C, *and*

(6) *if* $A \neq 0$ *then there is a natural injection* $f_*\mathcal{O}_A \hookrightarrow f_*\mathcal{O}_A(B|_A)$.

Proof: As usual, for a sheaf $\mathcal{F}$ let $\mathcal{F}(U)$ denote its sections over an open set U, see [Har77, Section II.1]. By definition, for any open $U \subset C$,

$$(f_*\mathcal{O}_Y(B))(U) = (\mathcal{O}_Y(B))(f^{-1}U).$$

Since B is vertical, there is a finite set $\Sigma \subset C$ such that $B \subset f^{-1}(\Sigma)$. Thus if $U \subset C \setminus \Sigma$ then $(f_*\mathcal{O}_Y(B))(U)$ consists of all regular functions on $f^{-1}(U)$. Since the fibers of f are geometrically connected and projective, every regular function is constant on them, thus the regular functions on $f^{-1}(U)$ are the pull backs of regular functions on U since Y is reduced.

If $U \cap \Sigma \neq \emptyset$, then, since B is effective, it is still true that the pull backs of regular functions on U are all in $(\mathcal{O}_Y(B))(f^{-1}U)$, but we may also have some rational functions which have poles along B. Thus $f^*\mathcal{O}_C \subset \mathcal{O}_Y(B)$ and $\mathcal{O}_Y(B) \subset f^*\mathcal{K}_C$ where $\mathcal{K}_C$ denotes the sheaf of all rational functions as in [Har77, p. 160]. Pushing these forward, we get injections

$$\mathcal{O}_C \hookrightarrow f_*\mathcal{O}_Y(B) \hookrightarrow \mathcal{K}_C.$$

As we saw, $\mathcal{O}_C \hookrightarrow f_*\mathcal{O}_Y(B)$ is an isomorphism over $C \setminus \Sigma$, thus $f_*\mathcal{O}_Y(B)$ is a torsion free coherent sheaf on C which has rank 1 on a dense open set, hence it is locally free of rank 1 everywhere. (Indeed, for any smooth curve C, its local rings $\mathcal{O}_{c,C}$ are principal ideal domains, hence every finitely generated torsion free $\mathcal{O}_{c,C}$–module is free. Therefore every torsion free coherent sheaf $\mathcal{F}$ on a connected smooth curve is locally free of rank r for some $r \geq 0$. Thus if a coherent sheaf $\mathcal{F}$ has rank 1 on a dense open subset of C then $\mathcal{F}$ is a rank 1 locally free sheaf on C.)

By construction B is an effective divisor which has no irreducible components in common with A, thus $B|_A$ is an effective Cartier divisor, which gives an injection $\mathcal{O}_A \hookrightarrow \mathcal{O}_A(B|_A)$ as in (7.6). Applying f_* we get another injection $f_*\mathcal{O}_A \hookrightarrow f_*\mathcal{O}_A(B|_A)$. ∎

Putting together (4), (5) and (6) of the above claims, we see that $f_*\mathcal{O}_A$ is the quotient of a subsheaf $\mathcal{M}$ of the rank 1 locally free sheaf $f_*\mathcal{O}_Y(B)$. Since C is a smooth curve, its local rings $\mathcal{O}_{c,C}$ are principal ideal domains, hence $\mathcal{M}$ itself is locally free of rank 1.

Finally, let $A(c)_1, \dots, A(c)_m$ be those connected components of A which are contained in $f^{-1}(c)$. We have surjections

$$\mathcal{M} \twoheadrightarrow f_*\mathcal{O}_A \twoheadrightarrow \sum_{i=1}^{m} H^0(A(c)_i, \mathcal{O}_{A(c)_i}),$$

which induces a surjection on the fibers over c. Since $\mathcal{M}$ is a rank 1 locally free sheaf, its fiber over c is k. On the other hand, each $H^0(A(c)_i, \mathcal{O}_{A(c)_i})$ contains at least the constant sections, thus we get a surjection $k \twoheadrightarrow k^n$ for some $n \geq m$. This implies that $m = n = 1$, hence $f|_A$ has geometrically connected fibers. ∎

Corollary 10: *Let Y be a smooth, projective variety over a field of characteristic 0, C a smooth curve and $f\colon Y \to C$ a dominant morphism with geometrically connected simple normal crossing fibers. Let $D = \sum a_i P_i$ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y such that*

(1) *D is f-vertical and*

(2) *$-(K_Y + D)$ is ample.*

Then, for every $c \in C$, the fiber F_c contains a $k(c)$-irreducible component which is geometrically irreducible.

Proof: Since D is f-vertical, its support is contained in the union of finitely many fibers of f, which are assumed to be simple normal crossing divisors. Thus

D itself is a simple normal crossing divisior and all the conditions of Theorem 9 are satisfied.

Given a closed point $c \in C$, let G be any divisor on C which is linearly equivalent to 0 such that $c \in \operatorname{Supp} G$. Note that f^*G is linearly equivalent to 0, thus we can add any rational multiple of f^*G to D without changing the assumptions of Theorem 9. Write $f^*G = \sum e_i P_i$. Then

$$D + \lambda f^*G = \sum (a_i + \lambda e_i) P_i.$$

Thus if we set

$$D' := D + \lambda_0 f^*G \quad \text{where } \lambda_0 := \min \left\{ \frac{1 - a_i}{e_i} : f(P_i) = c \right\},$$

then we have achieved that

(3) in a neighborhood of F_c, every irreducible component of D' has coefficient ≤ 1, and

(4) at least one irreducible component of F_c has coefficient 1 in D'.

Let $E \subset F_c$ be such a component. We claim that E is geometrically irreducible.

To see this, let m be the multiplicity of E in f^*G and consider

$$D'' := D' - (\epsilon/m) f^*G + \epsilon E \quad \text{for } 0 < \epsilon \ll 1.$$

This choice assures that

(5) in a neighborhood of F_c, every irreducible component of D'' has coefficient ≤ 1,

(6) there is only one irreducible component $E \subset F_c$ which has coefficient 1 in D'', and

(7) the $\mathbb{Q}$-divisor $-(K_Y + D'') \sim_{\mathbb{Q}} -(K_Y + D) - \epsilon E$ is ample for $0 < \epsilon \ll 1$ since ampleness is an open condition by Kleiman's criterion [Kl366, p. 325, Theorem 1].

Thus in a neighborhood of F_c, $E = D''_{\geq 1}$ and so, by Theorem 9, E is geometrically connected. Since E is smooth by assumption it is also geometrically irreducible. ∎

Remark 11: The assumption in Corollary 10 (1) can be relaxed considerably. The proof works without changes if Corollary 10 (1) is replaced by the following three conditions:

(1.i) $D_{\geq 1}$ is f-vertical,

(1.ii) $\operatorname{Supp} D +$ (any fiber of f) has simple normal crossings, and

(1.iii) $H^0(F_{gen}, \mathcal{O}_{F_{gen}}(D^*)$ is 1-dimensional over $k(C)$, where

$$D^* := -\sum_{i:a_i<0} [a_i]P_i$$

and F_{gen} is the generic fiber of f. (Here $[b]$ denotes the integral part of a number b.)

This more general case is useful since it can be used to prove that Theorem 3 holds even when the generic fiber is a $\mathbb{Q}$-Fano variety.

We need a further technical lemma.

Lemma 12: *Notation and assumptions as in Theorem 3. Then there is a closed subvariety $V \subset \mathbb{P}^N \times C$ with projection $\pi : V \to C$, an ample divisor L on $\mathbb{P}^N \times C$ and an open subset $C^0 \subset C$ such that*

(1) $V^0 := \pi^{-1}(C^0) \cong g^{-1}(C^0)$ *and*
(2) $-mK_{V^0} \sim L|_{V^0}$ *for some* $m > 0$.

Proof: I first explain the proof for Z as constructed in Paragraph 5. In this case we can take $V := Z$, $\pi := g$ and $N = n$. Let π_C, π_P be the the coordinate projections of $\mathbb{P}^N \times C$ and H a hyperplane on $\mathbb{P}^N$. Set $H_i := (uf_i + vh_i = 0) \subset \mathbb{P}^n \times \mathbb{P}^1$. By generic smoothness (cf. [Har77, Section III.10.7]), there is an open subset $C^0 \subset C = \mathbb{P}^1$ such that

(3) the projections $H_1 \cap \cdots \cap H_j \to \mathbb{P}^1$ are smooth over C^0 with fiber dimension $N - j$ for $j = 1, \ldots, s$.

Repeatedly applying the adjunction formula [Har77, Section II.8.20], we obtain that

$$\begin{aligned} -K_{V^0} &\sim ((N+1)\pi_P^* H - \sum H_i)|_{V^0} \\ &\sim (N + 1 - \sum \deg f_i)(\pi_P^* H)|_{V^0}. \end{aligned}$$

Let B be any effective divisor whose support is contained in $C \setminus C^0$. Since B is ample on C and $(N + 1 - \sum \deg f_i)H$ is ample on $\mathbb{P}^N$, we conclude that $L := \pi_C^* B + (N+1-\sum \deg f_i)\pi_P^* H$ is ample on $\mathbb{P}^N \times C$ (cf. [Har77, Exercise II.5.11].) By construction, $-K_{V^0} \sim L|_{V^0}$.

The general case is very similar. We can choose an open subset $C^0 \subset C$ such that the fibers F_c of g over C^0 are all smooth, $-mK_{F_c}$ is very ample for some $m > 0$ and $(g_* \mathcal{O}_Z(-mK_Z))|_{C^0}$ is free of rank $N+1$ for some N. This defines an embedding $g^{-1}(C^0) \hookrightarrow \mathbb{P}^N \times C^0$. Let $V \subset \mathbb{P}^N \times C$ be the closure of its image. We take $L := \pi_C^* B + \pi_P^* H$, then $-mK_{V^0} \sim L|_{V^0}$. ∎

13 Proof of Theorem 3. Notation as above. We can apply Hironaka's theorem on resolution of singularities [Hir64, p. 132, Theorem 1] to get $h_1: V_1 \to V$ such that V_1 is smooth and the isomorphism $\pi^{-1}(C^0) \cong g^{-1}(C^0)$ lifts to a morphism $\phi_1: V_1 \to Z$ [Hir64, p. 144, Paragraph 1].

Let $F^{sing} \subset V_1$ be the union of all singular fibers of $\pi \circ h_1$. Applying Hironaka's theorem on resolution of subschemes [Hir64, p. 146, Corollary 3] to $F^{sing} \subset V_1$ we get $h_2: V_2 \to V_1$ and $h := h_1 \circ h_2: V_2 \to V$ such that every fiber of $\pi \circ h: V_2 \to C$ is a simple normal crossing divisor. In particular, for every $c \in C$, every $k(c)$-irreducible component of the reduced fiber $\operatorname{red} F_c$ is smooth.

Throughout these resolutions we do not blow up anything above C^0.

Since $h: V_2 \to V$ is a composite of blow ups of subvarieties, there is an $m_2 > 0$ and an h-exceptional divisor E such that $m_2 h^*(L|_V) - E$ is ample on V_2 (cf. [Har77, Propositions II.7.10.b and II.7.13]). Dividing by $m \cdot m_2$ we conclude that there is an ample $\mathbb{Q}$-divisor M on V_2 such that

$$-K_{V_2}|_{V^0} \sim_{\mathbb{Q}} M|_{V^0}.$$

Thus there is a $\mathbb{Q}$-divisor D supported in $V_2 \setminus V^0$ such that

$$-(K_{V_2} + D) \sim_{\mathbb{Q}} M.$$

Since the support of D is contained in a union of fibers of $\pi \circ h$, it is a simple normal crossing divisor. Thus D is vertical and the Corollary 10 (1) and (2) hold.

Hence by Corollary 10, every fiber of $\pi \circ h: V_2 \to C$ contains a geometrically irreducible component.

Since every fiber of $g: Z \to C$ is dominated by a fiber of $\pi \circ h: V_2 \to C$, we conclude that every fiber of $g: Z \to C$ contains a geometrically irreducible subvariety.

Finally, assume that every $k(c)$-irreducible component of $g^{-1}(c)$ is smooth (or normal). Let $W \subset g^{-1}(c)$ be a geometrically irreducible subvariety and $F \subset g^{-1}(c)$ an irreducible component containing W. Write $F \times_{k(c)} \bar{k} = F_1 + \cdots + F_m$ where the F_i are irreducible over $\bar{k}$. One of the F_i contains $W_{\bar{k}}$, but then so do all the others since the F_i are conjugate over k. Since $F \times_{k(c)} \bar{k}$ is normal, this implies that $m = 1$ and F is geometrically irreducible over $k(c)$. ∎

Theorem 3 naturally raises the following question:

QUESTION 14: *Which "natural" classes of schemes $\mathbb{S}$ satisfy the following property*

$$(*)\qquad \begin{array}{c}\text{For every field } k \text{ and for every } k\text{-scheme } S \in \mathbb{S},\\ S \text{ contains a geometrically irreducible subscheme.}\end{array}$$

We have shown that $(*)$ holds for

$$\mathbb{S} = \{\text{degenerations of Fano varieties}\}.$$

There are two immediate generalizations, but $(*)$ fails for both. First, degenerations of Fano varieties are all rationally chain connected, that is, any two $\bar{k}$-points can be connected by a chain of rational curves over $\bar{k}$. (See [Kol96, Chapter IV] for a general overview.)

The triangle $(xyz = 0) \subset \mathbb{P}^2$ is rationally chain connected. Let $K/\mathbb{Q}$ be any cubic extension with norm form $N(x, y, z)$. Then $C_N := (N(x, y, z) = 0)$ has no geometrically irreducible $\mathbb{Q}$-subvarieties but it is isomorphic to the triangle over $\bar{\mathbb{Q}}$.

One can also try to work with singular Fano schemes. That is, schemes X such that ω_X is a line bundle such that ω_X^{-1} is ample. Here $(*)$ again fails.

Take the affine variety $(N(x, y, z)+x^4+y^4+z^4 = 0) \subset \mathbb{A}^3$. Blow up the origin to get Y. The exceptional curve is isomorphic to C_N, let I be its ideal sheaf. Then $X = \operatorname{Spec}_Y \mathcal{O}_Y/I^2$ is a Fano scheme with no geometrically irreducible $\mathbb{Q}$-subvarieties.

I have, however, no counterexamples to the following questions:

QUESTION 15: *Does $(*)$ hold for the following two classes of schemes:*
1. *Degenerations of smooth rationally connected varieties.*
2. *Reduced Fano schemes.*

The recent preprint [Sta06] proves (1) in case k contains an algebraically closed field.

In fact, in both cases it may be true that such a scheme contains a geometrically irreducible component which is also rationally connected:

QUESTION 16: *Let k be a field of characteristic 0, C a smooth k-curve, Z a smooth k-variety and $g\colon Z \to C$ a projective morphism. Assume that*
1. *the generic fiber F_{gen} is rationally connected,*
2. *every fiber is a simple normal crossing divisor (in particular, every $k(c)$-irreducible component of $g^{-1}(c)$ is smooth).*

Is it true that every fiber $g^{-1}(c)$ contains a $k(c)$-irreducible component which is rationally connected (and hence geometrically irreducible)?

Remark 17 (Positive characteristic): The conjecture of Ax needs only minor modifications in positive characteristic, see [FJ05, Chapter 21].

It is known that for any prime p, the following are equivalent:

(1) $\mathbb{F}_p(t)$ is weakly C_1.

(2) Every field of characteristic p is weakly C_1.

(3) Every perfect PAC field of characteristic p is C_1.

This is still not known but, as Jarden pointed out, Theorem 1 implies that for any fixed n there is a $p(n)$ such that if K is a perfect PAC field of characteristic $p \geq p(n)$ and $f(x_0, \dots, x_n)$ is homogeneous of degree $\leq n$ then it has a nontrivial zero in K.

Note also that every perfect PAC field of characteristic p is C_2 [FJ05, Theorem 21.3.6(b)].

The proof in this note has difficulties in positive characteristic. First, resolution of singularities is not known (but it is expected to be true). Second, Kodaira's vanishing theorem and its generalization (8) are false in positive characteristic. As far as I know, however, all versions of the Kollár–Shokurov Connectedness Theorem may hold in positive characteristic.

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Added in proof. Question 16 has been solved in Amit Hogadi and Chenyang Xu, Degenerations of Rationally Connected Varieties, 2006, math.AG/0606666.

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